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Matrix models, Argyres-Douglas singularities and double scaling limits

Gaetano Bertoldi*
Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge MA 02139, USA
E-mail: bertoldi@mit.edu

Abstract: We construct an $\mathcal{N}=1$ theory with gauge group $\mathrm{U}(nN)$ and degree $n+1$ tree level superpotential whose matrix model spectral curve develops an Argyres-Douglas singularity. The calculation of the tension of domain walls in the $\mathrm{U}(nN)$ theory shows that the standard large-N expansion breaks down at the Argyres-Douglas points, with tension that scales as a fractional power of N. Nevertheless, it is possible to define appropriate double scaling limits which are conjectured to yield the tension of 2-branes in the resulting $\mathcal{N}=1$ four dimensional non-critical string theories as proposed by Ferrari.

Keywords: 1/N Expansion, Matrix Models.

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Contents

1. Introduction

In [1, 2, 3], Dijkgraaf and Vafa conjectured that the exact superpotential and gauge couplings of a class of $\mathcal{N} = 1$ super Yang-Mills theories can be calculated by doing perturbative computations in an auxiliary matrix model. They considered theories with a polynomial superpotential $\mathcal{W}(\Phi)$ for the chiral adjoint field Φ and proposed that $\mathcal{W}$ is actually the potential in the related matrix model. Furthermore, only planar diagrams in the matrix model contribute to the effective superpotential. This striking result was later proved with perturbative field theory arguments in [4] and by the analysis of the generalized Konishi anomaly in [5]. The solution of the matrix model in the planar limit is captured by the so-called spectral curve, which is given by

$$y^2 = \mathcal{W}_n'(x)^2 + f_{n-1}(x)\,, \tag{1.1}$$

where n is the degree of $\mathcal{W}'(x)$ and $f_{n-1}(x)$ is a polynomial of degree $n-1$. The above curve is a hyperelliptic Riemann surface of genus $n-1$, a double cover of the x plane with $2n$ branch points. The values of the glueball superfields S_k and the expression of the effective superpotential are related to integrals of the meromorphic one-form $y\,dx$ over the curve (1.1).

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In [6], Ferrari studied an $\mathcal{N}=1$ U(N) gauge theory with cubic superpotential and using the results of [7] discovered that, in the phase where the gauge group is unbroken, there are critical values of the superpotential couplings where the effective superpotential is non-analytic and the standard large-N expansion is singular. In fact, the tension of supersymmetric domain walls scales as a fractional power of N at the critical points. This breakdown of the $1/N$ expansion can be compensated by taking the limit $N \to \infty$ and approaching the critical points in a correlated way. Furthermore, these double scaling limits are conjectured to define an $\mathcal{N}=1$ four dimensional non-critical string theory. This relies on a proposal made by Ferrari on how to generalize the old matrix model approach to non-critical strings [8] to the four dimensional case [9]–[13]. In this approach the matrix model integral is replaced by the four dimensional gauge theory path integral itself and the continuous worldsheets are built out of the large Feynman diagrams in the $N \to \infty$ limit. It was also shown in [6] that, from the matrix model point of view, the singularity corresponds to a transition from a two-cut solution to a one-cut solution. A cycle of the genus 1 spectral curve that describes the two-cut solution shrinks to zero size.

Ferrari also raised the question of the structure of higher order critical points à la Argyres-Douglas [14]. A matrix model spectral curve undergoes such a degeneration when two or more cycles with *non-vanishing intersection* shrink to zero size simultaneously. These singularities were first investigated in the context of $\mathcal{N}=2$ super Yang-Mills theories, whose low-energy physics is encoded by Seiberg-Witten hyperelliptic curves [15, 16, 17]. Their importance lies in the fact that, since the vanishing cycles have non-trivial intersection, the low-energy theory contains both electric and magnetic charges [14]. Furthemore, they are non-trivial interacting $\mathcal{N}=2$ conformal field theories [14, 18, 19, 20] and they provide the first quantitative check of the scenario advocated by Ferrari. In fact, it was shown in [10, 11], that these $\mathcal{N}=2$ CFTs can be used to define double scaling limits that yield dual non-critical string theories.

In this paper, higher critical points à la Argyres-Douglas in $\mathcal{N}=1$ theories are constructed and studied. In particular, a U(nN) gauge theory breaking to U$(N)^n$ in the presence of the one-parameter superpotential

$$\mathcal{W}(\Phi) = g_n \left(\frac{1}{n+1}\, \Phi^{n+1} - u\, \Phi \right), \qquad n \geq 3\,, \tag{1.2}$$

is analysed in detail. There are two values of the parameter u where the spectral curve develops an Argyres-Douglas singularity. The values of the glueball superfields S_k, $k = 1, \ldots n$, are calculated exactly by first showing that they are solutions of a linear second-order differential equation in u and then evaluating them in the semiclassical limit. This differential equation is the Picard-Fuchs equation for the periods of the meromorphic one-form ydx on the spectral curve. Then, the exact expression of the effective superpotential is derived and is shown to be non-analytic at the critical points. As a consequence, the large-N expansion breaks down at the Argyres-Douglas points with the tension of domain walls scaling as a fractional power of N. However, as in [6], there exists a double scaling limit

$$x \to 1\,, \qquad N \to \infty\,, \qquad N(1-x) = cnst = \frac{1}{\kappa}\,, \qquad x = \frac{4\Lambda_0^{2n}}{u^2}\,, \tag{1.3}$$

which is conjectured to yield a non-perturbative definition of an $\mathcal{N}=1$ four dimensional non-critical string theory.

2. Argyres-Douglas singularities

Argyres-Douglas singularities were originally investigated in the context of $\mathcal{N}=2$ super Yang-Mills theories, whose low-energy physics is encoded by Seiberg-Witten hyperelliptic curves [15, 16, 17]. The fact that the vanishing cycles have non-trivial intersection implies that the low-energy $\mathcal{N}=2$ theory has massless solitons with *mutually non-local* charges [14]. Namely, these solitons are both electrically and magnetically charged under the same U(1) factor. The theories at these points are actually superconformal [18]. As an illustration, let us consider the $\mathcal{N}=2$ Seiberg-Witten curve for SU(3) [15, 16, 17]

$$y^2 = (x^3 - ux - v)^2 - 4\Lambda^6 .$$

The above genus 2 hyperelliptic curve is singular whenever the polynomial on the r.h.s. has at least a double root, which is equivalent to the vanishing of its discriminant

$$\Delta = 2^{12}\Lambda^{18}\left(4u^3 - 27(v+2\Lambda^3)^2\right)\left(4u^3 - 27(v-2\Lambda^3)^2\right) .$$

For instance for $v=0$, $u = 3\,e^{2\pi ik/3}\Lambda^2$, $k=0,1,2$, the curve reduces to

$$y^2 = (x - e^{\pi ik/3}\Lambda)^2(x + e^{\pi ik/3}\Lambda)^2(x^2 - 4e^{2\pi ik/3}\Lambda^2) ,$$

which has two double roots. This is the limit where two *mutually local* dyons become massless. Argyres-Douglas points in moduli space, however, correspond to higher order singularities. An example is given by $u=0$ and $v=\pm 2\Lambda^3$, where the curve becomes [14]

$$y^2 = x^3(x^3 - 2v) , \tag{2.1}$$

and correspondingly two cycles with non-vanishing intersection shrink to zero size.

In the following, a simple generalization of the above singularity will be considered. In particular, the on shell spectral curve of the $\mathcal{N}=1$ system studied in the paper is going to be

$$y^2 = (x^n - u)^2 - 4\Lambda^{2n} , \qquad n \geq 3 .$$

It is easy to recognize that for $u = \pm 2\Lambda^n$, n out of the $2n$ branch points coalesce leading to an Argyres-douglas singularity. Before introducing the specific model which is object of study, it is necessary to review the strong coupling approach to the study of $\mathcal{N}=1$ gauge theories with polynomial superpotentials $\mathcal{W}(\Phi)$ [21, 22].

3. The strong coupling approach

The dynamics of $\mathcal{N}=1$ U(N) gauge theories with polynomial superpotentials can be studied by treating $\mathcal{W}(\Phi)$ as a perturbation of the underlying strongly coupled gauge

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theory with $\mathcal{W} = 0$. The latter system has $\mathcal{N} = 2$ supersymmetry and a Coulomb moduli space of vacua described by a Seiberg-Witten curve [15, 16, 17]

$$y^2 = P_N^2(x) - 4\Lambda^{2N} ,$$

where the coefficients of N-th order polynomial $P_N(x)$ depend on the N moduli $\langle tr\Phi^r \rangle$, $r = 1, \ldots, N$. In this strong coupling approach, which was developed in [21] using the methods of [22], $\mathcal{W}$ is regarded as an effective superpotential on the moduli space. The generic low energy group on the Coulomb moduli space is $\mathrm{U}(1)^N$. Vacua in which the low energy group of the $\mathcal{N} = 1$ theory is $\mathrm{U}(1)^n$, for $n < N$, can be found by extremizing the superpotential on submanifolds of the Coulomb branch where $N - n$ monopoles of the $\mathcal{N} = 2$ theory are massless. The superpotential lifts all of the moduli space except for a finite set of vacua. At points where $N - n$ mutually local monopoles become massless, the Seiberg-Witten curve has the following factorization

$$y^2 = P_N(x)^2 - 4\Lambda^{2N} = F_{2n}(x) H_{N-n}^2(x) , \tag{3.1}$$

where the polynomials on the r.h.s. have simple roots. This factorization is satisfied on an n-dimensional submanifold of the Coulomb moduli space on which the superpotential should be extremized in order to find the $\mathcal{N} = 1$ vacua. In [21], Cachazo, Intriligator and Vafa showed that this yields an on shell relation between the tree level superpotential and the polynomial $F_{2n}(x)$. In particular, when the degree of $\mathcal{W}'(x)$ is equal to n, the highest $n+1$ coefficients of $F_{2n}(x)$ are given in terms of $\mathcal{W}'(x)$ as follows

$$F_{2n}(x) = \frac{1}{g_n^2} \mathcal{W}'(x)^2 + \mathcal{O}(x^{n-1}) . \tag{3.2}$$

Given $\mathcal{W}$, the above relation determines $F_{2n}(x)$ in terms of n unknown coefficients that are fixed by requiring the existence of a polynomial $H_{N-n}(x)$ such that the factorization (3.1) holds. This determines $P_N(x)$ or equivalently the $\mathcal{N} = 2$ vacuum.

Conversely, given a polynomial $P_N(x)$ with the above factorization, one may look for a superpotential consistent with this vacuum. This inverse technique was used in [23] to rederive the $\mathcal{N} = 2$ solution using the geometric engineering approach of [21]. It was also used in [24] to study the various phases of such $\mathcal{N} = 1$ gauge theories and the structure of their parameter space. The same authors also provided a generalization of the above approach to include the cases $deg\, \mathcal{W}(\Phi) > n + 1$ and $deg\, \mathcal{W}(\Phi) > N$. Therefore, in order to construct examples of matrix model spectral curves which develop Argyres-Douglas singularities, one can start from a p-parameter family of $\mathcal{N} = 2$ hyperelliptic curves that displays such a degeneration in some appropriate limit. Then, using the inverse technique, it is possible to determine a superpotential consistent with these curves. The procedure yields the corresponding on shell family of $\mathcal{N} = 1$ spectral curves and once this is given one can study the gauge theory along the lines of [21, 1, 2, 3]. In the following, a one-parameter family of genus $n - 1$ hyperelliptic curves that can develop Argyres-Douglas singularities is introduced. Then, a consistent order $(n + 1)$ superpotential $\mathcal{W}(\Phi)$ is determined. Finally, the effective superpotential and glueball superfields S_k will be evaluated.

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3.1 The model

Consider a U(n) Seiberg-Witten curve of the following form

$$y^2 = P_n(x)^2 - 4\Lambda^{2n} = (\, x^n - u\,)^2 - 4\Lambda^{2n}\,. \tag{3.3}$$

The above curve has genus $n-1$ and is singular whenever the discriminant Δ of the polynomial on the right hand side of (3.3) vanishes

$$\Delta = (2n)^{2n}(-4\Lambda^{2n})^n(u^2 - 4\Lambda^{2n})^{n-1}\,. \tag{3.4}$$

In particular, in the limit $u \to \pm 2\Lambda^n$, n branch points collide and the curve reduces to

$$y^2 = x^n\,(\, x^n \mp 4\Lambda^n\,)\,.$$

For $n \geq 3$, these are Argyres-Douglas singularities. Note that the curve (3.3) depends on one parameter only, u, and that it has a $\mathbb{Z}_n$ symmetry generated by

$$(x, y) \to \left(e^{2\pi i/n} x, y\right). \tag{3.5}$$

In this case the factorization of the Seiberg-Witten curve is trivial, namely for a generic value of u there are no double roots

$$y^2 = P_n^2(x) - 4\Lambda^{2n} = F_{2n}(x)\,. \tag{3.6}$$

Then, the low energy group of the above theory is $\mathrm{U}(1)^n$ and the degree of a polynomial superpotential $\mathcal{W}(\Phi)$ consistent with the above Seiberg-Witten curve has to be at least $n+1$. When the degree of $\mathcal{W}(\Phi)$ is equal to $n+1$, the matrix model spectral curve actually coincides with the Seiberg-Witten curve [23]

$$y^2 = P_n^2(x) - 4\Lambda^{2n} = F_{2n}(x) = \frac{1}{g_n^2}\left(\mathcal{W}_n'(x)^2 + f_{n-1}(x)\right)\,, \tag{3.7}$$

where

$$\mathcal{W}_n'(x) = g_n P_n(x)\,, \qquad f_{n-1}(x) = -4\, g_n^2 \Lambda^{2n}\,. \tag{3.8}$$

Then, modulo the addition of a constant, the tree level superpotential is

$$\mathcal{W}(\Phi) = \frac{g_n}{n+1}\,\Phi^{n+1} - g_n u\,\Phi\,. \tag{3.9}$$

When $u = 0$, the field Φ becomes critical and there is a singularity in the classical space of parameters. Note also that, since $f_{n-1}(x)$ is constant, the total glueball superfield $S = \sum_{k=1}^n S_k$ vanishes identically [23, 5]. Finally, by the *multiplication map* introduced in [21], the above U(n) theory can be mapped to a U(nN) theory with the same superpotential. In the remainder of the paper we are going to set $g_n = 1$.

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4. The glueball superfields

The glueball superfields S_k are given by the period integrals of the meromorphic one-form ydx along the closed loops A_k surrounding the k-th branch cut [21, 1, 2, 3]

$$S_k = \frac{1}{2\pi i}\oint_{A_k} y\,dx = \frac{1}{2\pi i}\oint_{A_k}\sqrt{(x^n-u)^2-4\Lambda^{2n}}\,dx\,. \tag{4.1}$$

We are going to find the exact expression of the above periods by first deriving a second-order linear differential equation in u satisfied by them, a so-called Picard-Fuchs equation. The evaluation of the semiclassical limit of S_k will then allow to fix the linear combination of the two independent solutions. A similar analysis was carried out in the context of $\mathcal{N}=2$ Seiberg-Witten theories in [25]. First of all

$$\partial_u S_k = \frac{1}{2\pi i}\oint_{A_k}\partial_u(\,y\,dx\,) = -\frac{1}{2\pi i}\oint_{A_k}\frac{(x^n-u)}{y}dx\,.$$

Taking another derivative we find

$$\partial_u^2 S_k = \frac{1}{2\pi i}\oint_{A_k}\partial_u^2(\,y\,dx\,) = -\frac{1}{2\pi i}\oint_{A_k}\frac{4\Lambda^{2n}}{y^3}dx\,.$$

By the following identity

$$\begin{aligned}\partial_u^2(\,y\,dx\,) &= \left(\frac{n-1}{n}\right)\frac{4\Lambda^{2n}}{4\Lambda^{2n}-u^2}\frac{dx}{y}+\frac{(x^n-u)u}{n(4\Lambda^{2n}-u^2)}\frac{dx}{y}+\partial_x\left(\frac{x(u^2-ux^n+4\Lambda^{2n})}{n(4\Lambda^{2n}-u^2)y}\right)dx\,,\\ &= -\left(\frac{n-1}{n^2}\right)\frac{1}{4\Lambda^{2n}-u^2}\Lambda\partial_\Lambda(ydx)-\frac{1}{n}\frac{u}{(4\Lambda^{2n}-u^2)}\partial_u(ydx)+\\ &\quad +\partial_x\left(\frac{x(u^2-ux^n+4\Lambda^{2n})}{n(4\Lambda^{2n}-u^2)y}\right)dx\,,\end{aligned}$$

we conclude that

$$\partial_u^2 S_k = \left(\frac{n-1}{n^2}\right)\frac{1}{u^2-4\Lambda^{2n}}\Lambda\partial_\Lambda S_k+\frac{1}{n}\frac{u}{(u^2-4\Lambda^{2n})}\partial_u S_k\,, \tag{4.2}$$

since the integral of an exact differential along a closed cycle is zero.

By (4.1), we can also see that S_k is a homogeneous function of degree $n+1$, which yields

$$(\Lambda\partial_\Lambda+nu\partial_u)\,S_k = (n+1)S_k\,. \tag{4.3}$$

Finally, by (4.2) and (4.3), we find

$$\left[\partial_u^2+\left(\frac{n-2}{n}\right)\frac{u}{u^2-4\Lambda^{2n}}\partial_u-\left(\frac{n^2-1}{n^2}\right)\frac{1}{u^2-4\Lambda^{2n}}\right]S_k = 0\,. \tag{4.4}$$

Note that every period of $y\,dx$ is a solution of the above Picard-Fuchs equation as long as the cycle is closed. In particular, the difference of any two derivatives of the prepotential $\mathcal{F}$ with respect to the glueball superfields is also a solution of (4.4).

4.1 Solution of the Picard-Fuchs equation

By the following change of variables

$$z = \frac{u^2}{4\Lambda^{2n}}\,.$$

Eq. (4.4) becomes a hypergeometric equation

$$\left[z(1-z)\partial_z^2 + \left(\frac{1}{2} - \frac{n-1}{n}z\right)\partial_z + \frac{n^2-1}{4n^2}\right] S_k = 0\,, \tag{4.5}$$

which is solved by [26]

$$C_{1,k}\, F\left(\frac{1}{2} - \frac{1}{2n}, -\frac{1}{2} - \frac{1}{2n}, \frac{1}{2}, \frac{u^2}{4\Lambda^{2n}}\right) + C_{2,k}\, u\, F\left(1 - \frac{1}{2n}, -\frac{1}{2n}, \frac{3}{2}, \frac{u^2}{4\Lambda^{2n}}\right). \tag{4.6}$$

In order to fix the coefficients $C_{1,k}$ and $C_{2,k}$, one can evaluate S_k in the semiclassical limit $\Lambda \to 0$, where the glueball superfield vanishes. Since $\lim_{\Lambda\to 0} z = \infty$, one needs to perform the analytic continuation of the above hypergeometric functions, which are defined as power series in the disc $|z| \leq 1$. Alternatively, one can rewrite the Picard-Fuchs equation in terms of a new variable that vanishes as $\Lambda \to 0$.

In the limit $u \to \infty$, which is dual to $\Lambda \to 0$, the solutions of eq. (4.4) are asymptotic to

$$u^{\alpha_\pm} f_\pm(u)\,,$$

where

$$\alpha_\pm = \frac{1}{n} \pm 1\,, \qquad \lim_{u\to\infty} f_\pm(u) \neq 0\,.$$

Setting $S_k = u^{\alpha_-} f_k(u)$ and changing variables to $z = \frac{4\Lambda^{2n}}{u^2}$, eq. (4.4) is equivalent to

$$\left[z(1-z)\partial_z^2 + \left(2 - \left(\frac{5}{2} - \frac{1}{n}\right) z\right)\partial_z - \frac{2n^2-3n+1}{4n^2}\right] f_k = 0\,, \tag{4.7}$$

which is again a hypergeometric equation. The condition that S_k vanishes in the limit $z = \frac{4\Lambda^{2n}}{u^2} \to 0$ implies that

$$S_k = C_{3,k}\, u^{-1+\frac{1}{n}}\, F\left(\frac{1}{2} - \frac{1}{2n}, 1 - \frac{1}{2n}, 2, \frac{4\Lambda^{2n}}{u^2}\right). \tag{4.8}$$

Note that the above expression is a power series in Λ^{2n}, namely S_k is given by an instanton sum. The value of $C_{3,k}$ can be found by evaluating the semiclassical limit of S_k more carefully

$$S_k = \frac{1}{2\pi i}\oint_{A_k} \sqrt{P_n^2 - 4\Lambda^{2n}}\, dx = -\frac{2\Lambda^{2n}}{2\pi i}\oint_{\gamma_k} \frac{1}{P_n(x)}\, dx + \mathcal{O}(\Lambda^{4n})\,, \tag{4.9}$$

where γ_k is a counterclockwise loop around the k-th root of $P_n(x) = x^n - u$,

$$x_k = e^{2\pi i k/n} u^{1/n}\,.$$

Then

$$-\frac{2\Lambda^{2n}}{2\pi i}\oint_{\gamma_k}\frac{1}{P_n(x)}\,dx \;=\; -2\Lambda^{2n}\frac{1}{P'(x_k)} \;=\; -\frac{2\Lambda^{2n}}{n}e^{2\pi i k/n}u^{-\frac{n-1}{n}}\,.$$

Finally, by (4.8), we find

$$C_{3,k} = -\frac{2\Lambda^{2n}}{n}e^{2\pi i k/n}\,, \tag{4.10}$$

and

$$S_k = -\frac{2\Lambda^{2n}}{n}e^{2\pi i k/n}u^{-1+1/n}\,F\left(\frac{1}{2}-\frac{1}{2n},1-\frac{1}{2n},2,\frac{4\Lambda^{2n}}{u^2}\right). \tag{4.11}$$

Note that this is consistent with the fact that

$$\sum_{k=1}^{n}S_k = 0\,,$$

since $\sum_{k=1}^{n}e^{2\pi i k/n} = 0$. We will actually show in a subsequent paragraph that $S_{k+1} = e^{2\pi i/n}S_k$ as a direct consequence of the symmetry (3.5). Furthermore, we will show that this symmetry also implies that $\frac{\partial\mathcal{F}}{\partial S_{k+1}} = e^{2\pi i/n}\frac{\partial\mathcal{F}}{\partial S_k}$.

Performing the analytic continuation of (4.6), which defines an absolutely convergent series for $|u| \le 2\Lambda^n$, to the region $|u| > 2\Lambda^n$, where (4.8) is valid, we are going to find relations among the various C_k's. The details are given in the appendix. The results are

$$C_{2,k} = -\frac{e^{i\pi/2}}{2\Lambda^n}\frac{\Gamma(1-1/2n)\Gamma(3/2+1/2n)}{\Gamma(1/2-1/2n)\Gamma(1+1/2n)}\,C_{1,k}\,, \tag{4.12}$$

and

$$C_{3,k} = e^{i\pi/2+i\pi/2n}\frac{2\pi}{\sin(\pi/n)}\left(\frac{\sqrt{\pi}\,2^{\frac{n-1}{n}}}{2n\Gamma(1+1/2n)\Gamma(-1/2-1/2n)}\right)\Lambda^{n-1}C_{1,k}\,. \tag{4.13}$$

4.2 Non-analytic behaviour close to the Argyres-Douglas points

By analytic continuation of (4.8)(4.11), we find [26]

$$\begin{aligned} S_k \;=\; C_{3,k}\,u^{-1+\frac{1}{n}}\Bigg(&A_1F\left(\frac{1}{2}-\frac{1}{2n},1-\frac{1}{2n},\frac{1}{2}-\frac{1}{n},1-\frac{4\Lambda^{2n}}{u^2}\right)+ \\ &+A_2\left(1-\frac{4\Lambda^{2n}}{u^2}\right)^{\frac{n+2}{2n}}F\left(\frac{3}{2}+\frac{1}{2n},1+\frac{1}{2n},\frac{3}{2}+\frac{1}{n},1-\frac{4\Lambda^{2n}}{u^2}\right)\Bigg)\,, \end{aligned} \tag{4.14}$$

which implies that close to the singularity at $u^2 = 4\Lambda^{2n}$

$$S_k \approx A_1 + A_2\left(\frac{u^2-4\Lambda^{2n}}{4\Lambda^{2n}}\right)^{\frac{n+2}{2n}} + \mathcal{O}(u^2-4\Lambda^{2n})\,, \tag{4.15}$$

where

$$A_1 = \frac{\Gamma(1/2+1/n)}{\Gamma(3/2+1/2n)\Gamma(1+1/2n)}\,,\qquad A_2 = \frac{\Gamma(-1/2-1/n)}{\Gamma(1/2-1/2n)\Gamma(1-1/2n)}\,. \tag{4.16}$$

Thus, we see that S_k is non-vanishing in the limit $u^2 \to 4\Lambda^{2n}$ and that it is non-analytic due to the fractional exponent $\frac{n+2}{2n}$, $(n \ge 3)$.

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5. The multiplication map

By the so-called *multiplication map* by N introduced in [21], the above $\mathrm{U}(n)$ theory with superpotential (3.9) can be mapped to a $\mathrm{U}(nN)$ theory with the same tree level superpotential. In particular, the vacuum considered up to now, which is a Coulomb vacuum with unbroken $\mathrm{U}(1)^n$ gauge group, is associated to N different vacua with unbroken $\mathrm{U}(N)^n$. In fact, given a set of polynomials $P_n(x)$, $F_{2m}(x)$ and $H_{n-m}(x)$, all with the highest coefficient equal to 1, that satisfy the following relations

$$P_n^2(x) - 4\Lambda^{2n} = F_{2m}(x) H_{n-m}^2(x)\,, \qquad F_{2m}(x) = \mathcal{W}'(x)^2 + f_{m-1}(x)\,, \tag{5.1}$$

it is possible to show that $\mathcal{Q}_{nN}(x) \equiv 2\Lambda_0^{nN}\eta^N \mathcal{T}_N\left(\frac{P_n(x)}{2\eta\Lambda_0^n}\right)$, where $\mathcal{T}_N$ is the N-th Chebishev polynomial of the first kind and η is a $2N$-th root of unity, is a polynomial of degree nN with highest coefficient equal to 1 and that it satisfies

$$\begin{aligned}
\mathcal{Q}_{nN}^2(x) - 4\Lambda_0^{2nN} &= 4\Lambda_0^{2nN}\left(\mathcal{T}_N\left(\frac{P_n(x)}{2\eta\Lambda_0^n}\right) - 1\right) \\
&= 4\Lambda_0^{2nN}\left[\left(\frac{P_n(x)}{2\eta\Lambda_0^n}\right)^2 - 1\right]\mathcal{U}_{N-1}^2\left(\frac{P_n(x)}{2\eta\Lambda_0^n}\right) \\
&= \left(P_n^2(x) - 4\eta^2\Lambda_0^{2n}\right)\left(\eta^{N-1}\Lambda_0^{n(N-1)}\mathcal{U}_{N-1}\left(\frac{P_n(x)}{2\eta\Lambda_0^n}\right)\right)^2,
\end{aligned}$$

where $\mathcal{U}_{N-1}$ is the $(N-1)$-th Chebishev polynomial of the second kind and we used the following relation

$$\mathcal{T}_N^2(x) - 1 = (x^2-1)\mathcal{U}_{N-1}^2(x)\,. \tag{5.2}$$

Therefore, if one sets

$$\Lambda^{2n} = \eta^2\Lambda_0^{2n}\,, \tag{5.3}$$

one finds that $\mathcal{Q}_{nN}(x)$ satisfies the following identity

$$\begin{aligned}
\mathcal{Q}_{nN}^2(x) - 4\Lambda_0^{2nN} &= F_{2m}(x) H_{n-m}^2(x)\left(\eta^{N-1}\Lambda_0^{n(N-1)}\mathcal{U}_{N-1}\left(\frac{P_n(x)}{2\eta\Lambda_0^n}\right)\right)^2 \\
&= \tilde{F}_{2m}(x)\tilde{H}_{nN-m}^2(x)\,,
\end{aligned} \tag{5.4}$$

where

$$\tilde{F}_{2m}(x) = F_{2m}(x)\,, \qquad \tilde{H}_{nN-m}(x) = H_{n-m}(x)\,\eta^{N-1}\Lambda_0^{n(N-1)}\mathcal{U}_{N-1}\left(\frac{P_n(x)}{2\eta\Lambda_0^n}\right) \tag{5.5}$$

As was explained in section 3, if one wants to find vacua of a $\mathrm{U}(nN)$ theory with low energy group $\mathrm{U}(1)^m$, one has to solve the following factorization problem

$$P_{nN}^2(x) - 4\Lambda_0^{2nN} = \tilde{F}_{2m}(x)\tilde{H}_{nN-m}^2(x)\,.$$

Then, eq. (5.4) implies that $P_{nN}(x) = \mathcal{Q}_{nN}(x)$ is a solution. Furthermore, since $\tilde{F}_{2m}(x) = F_{2m}(x)$, the vacua of the $\mathrm{U}(nN)$ theory have the same superpotential as the vacua of the

$\mathrm{U}(n)$ theory. It was also shown in [21], that a classical limit with unbroken $\prod_{j=1}^{k} \mathrm{U}(n_j)$ is mapped to a classical limit with unbroken $\prod_{j=1}^{k} \mathrm{U}(Nn_j)$. Finally, eq. (5.3) implies that for each vacuum of the $\mathrm{U}(n)$ theory one has N vacua of the $\mathrm{U}(nN)$ theory.

By eqs. (3.7), (5.3) and (5.5), the spectral curve relative to one of the N vacua of the $\mathrm{U}(nN)$ theory breaking to $\mathrm{U}(N)^n$ is given by

$$y^2 = (x^n - u)^2 - 4\eta^2\Lambda_0^{2n} . \tag{5.6}$$

This implies that the glueball superfields S_k, $k = 1, \dots, n$ of each $\mathrm{U}(N)$ factor are given by

$$S_k = -\frac{2\eta^2\Lambda_0^{2n}}{n} e^{2\pi i k/n} u^{-1+1/n} \, F\left(\frac{1}{2} - \frac{1}{2n}, 1 - \frac{1}{2n}, 2, \frac{4\eta^2\Lambda_0^{2n}}{u^2}\right) . \tag{5.7}$$

6. The effective superpotential

In this section, we are going to evaluate the effective superpotential for each of the N vacua of the $\mathrm{U}(nN)$ theory considered above. The supersymmetric domain walls interpolating between two such vacua are interesting objects to study and their tension can be calculated exactly once the expression of $\mathcal{W}_{\text{eff}}$ is known [28]. In section 7, the large-N limit of the tension will be considered. It follows that, due to the non-analyticity of $\mathcal{W}_{\text{eff}}$ at the Argyres-Douglas points, the standard $1/N$ expansion breaks down at the singularities. However, there exist double scaling limits of the tension that yield well-defined results. The general matrix model formula for the effective superpotential of a $\mathrm{U}(N)$ theory breaking to $\prod_{k=1}^{n} \mathrm{U}(N_k)$ is [24, 21, 1, 2, 3]

$$\mathcal{W}_{\text{eff}}(S) = \sum_{k=1}^{n} N_k \frac{\partial \mathcal{F}}{\partial S_k} + 2\pi i \tau_0 \sum_{k=1}^{n} S_k + 2\pi i \sum_{k=2}^{n} b_k S_k , \tag{6.1}$$

where the constants b_k are actually integers [24] and are related to periods of the one-form

$$T(x) = \left\langle Tr \frac{dx}{x - \Phi} \right\rangle , \tag{6.2}$$

the generating function of $Tr\, \Phi^k$, introduced in [5]. The term proportional to b_k in (6.1) is an imaginary multiple of S_k and thus represents a shift in the theta angle. In particular, relative to the theta angle of $\mathrm{SU}(N_1)$, the theta angle in $\mathrm{SU}(N_k)$ is greater by $2\pi b_k$, since we set $b_1 = 0$. As explained in [24], this implies that different mixtures of 't Hooft and Wilson loops, H and W, condense in $\mathrm{SU}(N_1)$ and $\mathrm{SU}(N_k)$. More precisely, if $W^{r_1}H$ condenses for $\mathrm{SU}(N_1)$ then $W^{r_k}H$ condenses for the other factor, where $r_k - r_1 = b_k$. These coefficients are necessary to determine how confinement is realized in a particular vacuum and will be determined below.

Let us note that

$$N_k = \frac{1}{2\pi i} \int_{A_k} T ,$$

and

$$\tau_0 = \frac{1}{2\pi i} \int_{B_1} \bar{T} , \qquad b_k = -\frac{1}{2\pi i} \int_{B_k} \bar{T} - \tau_0 ,$$

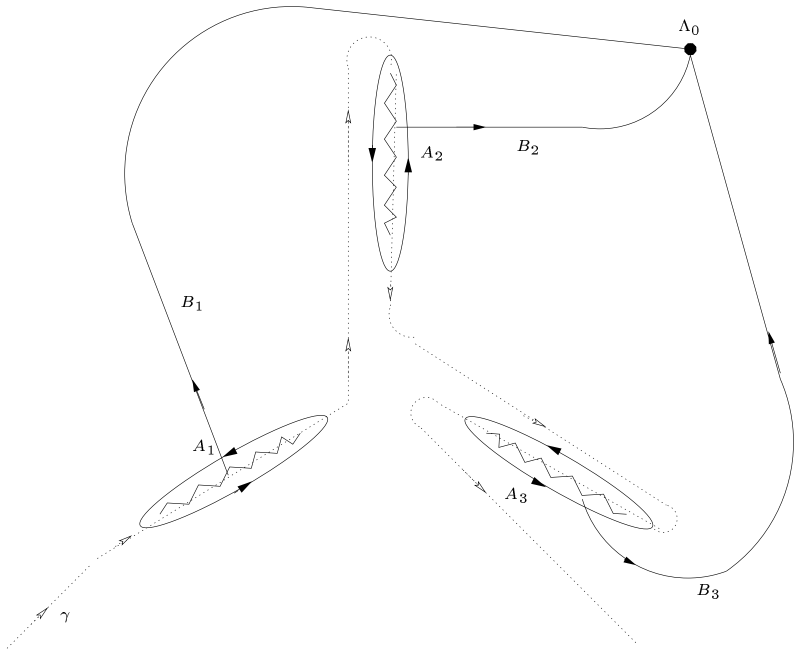

Figure 1: The B_k cycles and path γ for $n = 3$, $(Im\, u = 0)$.

where the A_k's are the closed cycles sorrounding the branch cuts of the spectral curve and the B_k's are the non-compact cycles connecting the points at infinity on the two sheets of the spectral curve passing through the k-th branch cut. The intersection pairings of these cycles are $A_j \cap A_k = B_j \cap B_k = 0$ and $A_j \cap B_k = \delta_{jk}$. In figure 1, we depicted the A and B paths on one of the two sheets for the case $n = 3$. By Λ_0 we denote the point at infinity. The path γ enters in the rigorous definition of the matrix model conjecture recently studied by Lazaroiu [27] involving holomorphic matrix models. By definition, γ threads the branch cuts and fixes the basis of A and B cycles on the spectral curve. The glueball superfields and the derivatives of the prepotential $\mathcal{F}$ are given by [21, 1, 2, 3]

$$S_k = \frac{1}{2\pi i} \oint_{A_k} y dx \,, \qquad \frac{\partial \mathcal{F}}{\partial S_k} = \int_{B_k} y dx \,. \tag{6.3}$$

Besides the closed cycles A_k it is useful to consider the closed cycles

$$C_k = B_{k+1} - B_k \,, \qquad k = 1, \ldots, n-1 \,, \tag{6.4}$$

which connect two neighbouring branch cuts [24], see figure 2. Then, by (6.3) and (6.4)

$$\frac{\partial \mathcal{F}}{\partial S_{k+1}} - \frac{\partial \mathcal{F}}{\partial S_k} = \oint_{C_k} y dx \,,$$

which immediately shows that the above difference is also a solution of the Picard-Fuchs equation (4.4). In fact, as we remarked above, every period of the meromorphic one-form ydx satisfies the Picard-Fuchs equation as long as the cycle is closed, which is the case of C_k.

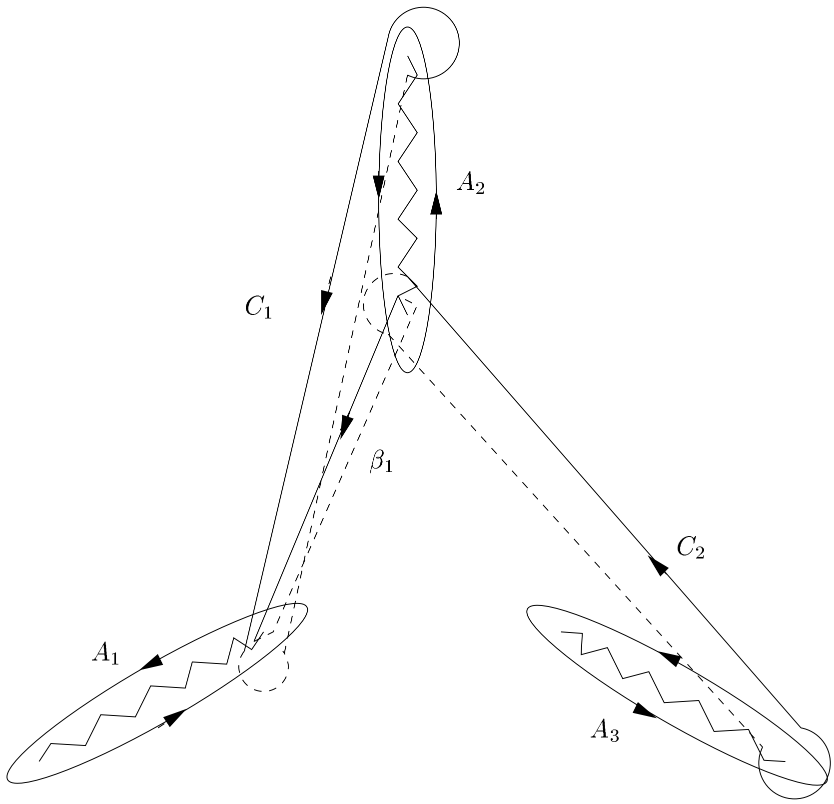

Figure 2: The A_k and C_l cycles for $n = 3$, $(Im\, u = 0)$.

Let us show that the symmetry (3.5) implies a simple relation between $\frac{\partial \mathcal{F}}{\partial S_k}$ and $\frac{\partial \mathcal{F}}{\partial S_{k-1}}$. First of all

$$\frac{\partial \mathcal{F}}{\partial S_k} = \int_{B_k} y dx = \lim_{\Lambda_0 \to \infty} 2 \int_{x_{k,+}}^{\Lambda_0} y dx \; - 2\,\mathcal{W}(\Lambda_0)\,,$$

where

$$x_{k,\pm} = e^{2\pi i k/n}\,(u \pm 2\Lambda^n)^{1/n}\;,$$

are the branch points of the spectral curve (3.3) and Λ_0 is the point at infinity on the upper sheet. Setting $x = e^{2\pi i/n}\,\tilde{x}$, the above integral becomes

$$\int_{x_{k,+}}^{\Lambda_0} y dx \;= \int_{x_{k,+}}^{\Lambda_0} \sqrt{(x^n - u)^2 - 4\Lambda^{2n}}\, dx \;= e^{2\pi i/n} \int_{x_{k-1,+}}^{\tilde{\Lambda}_0} \sqrt{(\tilde{x}^n - u)^2 - 4\Lambda^{2n}}\, d\tilde{x}\,,$$

where $\tilde{\Lambda}_0 = e^{-2\pi i/n}\Lambda_0$. Then

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial S_k} = \lim_{\Lambda_0 \to \infty} 2 \int_{x_{k,+}}^{\Lambda_0} y dx \; - 2\,\mathcal{W}(\Lambda_0) \;&=\; \lim_{\Lambda_0 \to \infty} 2\, e^{2\pi i/n} \int_{x_{k-1,+}}^{\tilde{\Lambda}_0} y dx \; - 2\, e^{2\pi i/n}\, \mathcal{W}(\tilde{\Lambda}_0) + \\ &+ \Big(2 e^{2\pi i/n}\, \mathcal{W}(\tilde{\Lambda}_0) - 2\,\mathcal{W}(\Lambda_0)\Big) = e^{2\pi i/n} \frac{\partial \mathcal{F}}{\partial S_{k-1}}\,,\end{aligned} \tag{6.5}$$

where we used

$$\mathcal{W}(x) = \frac{1}{n+1}\, x^{n+1} - u\, x\,. \tag{6.6}$$

Note that in our case the low-energy gauge group is $\mathrm{U}(1)^n$, namely $N_k = 1$, $k = 1, \ldots, n$. Then, it follows that

$$\sum_{k=1}^{n} N_k \frac{\partial \mathcal{F}}{\partial S_k} = \sum_{k=1}^{n} e^{2\pi i(k-1)/n} \frac{\partial \mathcal{F}}{\partial S_1} = 0\,, \tag{6.7}$$

and the expression of the effective superpotential reduces to

$$\mathcal{W}_{\text{eff}}(S) = 2\pi i \sum_{k=1}^{n} b_k S_k\,, \tag{6.8}$$

where we also used the fact that $\sum_{k=1}^{n} S_k = 0$. Similarly, we can show that $S_k = e^{2\pi i/n} S_{k-1}$. In fact

$$S_k = \frac{1}{2\pi i} \oint_{A_k} y dx = \frac{2}{2\pi i} \int_{x_{k,-}}^{x_{k,+}} y dx\,,$$

and by the same change of variable as before, namely $x = e^{2\pi i/n}\,\tilde{x}$, we find

$$S_k = \frac{2}{2\pi i}\, e^{2\pi i/n} \int_{x_{k-1,-}}^{x_{k-1,+}} y d\tilde{x} \; = e^{2\pi i/n} S_{k-1}\,. \tag{6.9}$$

6.1 The periods of T

In order to evaluate the effective superpotential (6.8), we need to evaluate the integer constants b_k or equivalently c_k

$$b_k = -\sum_{j=1}^{k-1} c_j\,, \quad (b_1 = 0)\,, \qquad c_j = \frac{1}{2\pi i} \oint_{C_j} T\,. \tag{6.10}$$

It was shown in [5] that the one-form T_n of the $\mathrm{U}(n)$ gauge group in the vacuum with unbroken $\mathrm{U}(1)^n$ and the above superpotential is given by

$$T_n = \frac{P_n'(x)\, dx}{\sqrt{P_n^2(x) - 4\Lambda^{2n}}} = d \log \left(P_n(x) + \sqrt{P_n^2(x) - 4\Lambda^{2n}} \right)\,. \tag{6.11}$$

Note that, since T_n is a logarithmic derivative, its period integrals are integer-valued. Furthermore, the transformation property of T_n under the multiplication by N map is very simple, namely [24]

$$T_{nN}(x) = N\, T_n(x)\,. \tag{6.12}$$

Therefore, it suffices to calculate the periods of T_n. First of all, since we are considering a $\mathrm{U}(n)$ theory which breaks to $\mathrm{U}(1)^n$, we have

$$N_k = \frac{1}{2\pi i} \oint_{A_k} T_n = 1\,. \tag{6.13}$$

In fact

$$N_k = \lim_{\Lambda \to 0} \frac{1}{2\pi i} \oint_{A_k} T_n = \frac{1}{2\pi i} \oint_{\gamma_k} \frac{P_n'(x)}{P_n(x)} dx = 1\,, \tag{6.14}$$

where γ_k is a counterclockwise loop around the k-th root of $P_n(x) = x^n - u$,

$$x_k = e^{2\pi i k/n} u^{1/n}\,.$$

To illustrate the calculation of the c_k's, let us consider the case $n = 3$ in more detail. In figure 2, the A_k cycles, $k = 1,2,3$, and C_l cycles, $l = 1,2$ are shown. Note that $C_1 = A_2 + \beta_1$. Furthermore, β_1 vanishes in the limit $u \to 2\Lambda^n$, since the "inner" branch points $x_{k,-} = e^{2\pi ik/n}(u - 2\Lambda^2)^{1/n}$, $k = 1,2,3$, collide. The crucial observation is that T has no poles in this particular limit, which yields

$$\lim_{u\to 2\Lambda^n} \oint_{\beta_1} T_n = 0\,.$$

Therefore, we can conclude that the integral of T along β_1 is zero and that

$$c_1 = \frac{1}{2\pi i}\oint_{C_1} T_n = \frac{1}{2\pi i}\oint_{A_2} T_n = 1\,. \tag{6.15}$$

Repeating the above argument, we can see that the same equality actually holds for any C_l and for general n, namely

$$c_k = 1\,, \qquad k = 1,\dots,n-1\,. \tag{6.16}$$

Then, by (6.9), (6.10), (6.12) and (6.16), the effective superpotential (6.8) of the $\mathrm{U}(nN)$ theory in the ℓ-th vacuum, $\ell = 1,\dots N$, is

$$\begin{aligned}\mathcal{W}_{eff,\ell}(S) &= 2\pi i\sum_{k=1}^{n} b_k S_k(u,\eta^{2\ell}\Lambda_0^{2n})\\ &= -2\pi i N\sum_{k=1}^{n}(k-1)e^{2\pi i(k-1)/n}S_1(u,\eta^{2\ell}\Lambda_0^{2n})\\ &= -2\pi i N\,\frac{n}{e^{2\pi i/n}-1}S_1(u,\eta^{2\ell}\Lambda_0^{2n})\,,\quad \eta = e^{\pi i/N}\,,\end{aligned} \tag{6.17}$$

where by (4.11) and (5.3)

$$S_1(u,\eta^{2\ell}\Lambda_0^{2n}) = -\frac{2\eta^{2\ell}\Lambda_0^{2n}}{n}e^{2\pi i/n}u^{-1+1/n}\,F\left(\frac{1}{2}-\frac{1}{2n},1-\frac{1}{2n},2,\frac{4\eta^{2\ell}\Lambda_0^{2n}}{u^2}\right). \tag{6.18}$$

Note that, by (4.15), the effective superpotential is non-analytic at the Argyres-Douglas points $u^2 = 4\Lambda^{2n}$.

7. The large-N limit

Following [6, 9], we expect a non-trivial behaviour of the large-N limit at the Argyres-Douglas critical points. An observable that can be determined exactly from the superpotential (6.17) is the tension of supersymmetric domain walls between the N vacua of the $\mathrm{U}(nN)$ theory. The tension of such a BPS domain wall interpolating between the p-th and q-th vacua, $p,q = 1,\dots,N$, is given by [28]

$$T_{p,q} = |\tau_{p,q}|\,, \tag{7.1}$$

where

$$\tau_{p,q} = N\left(\mathcal{W}_{eff,p}(S) - \mathcal{W}_{eff,q}(S)\right)\,. \tag{7.2}$$

The factor of N arises because we are considering the 't Hooft limit $N \to \infty$, $g_{YM}^2 \to 0$, with $\lambda = g_{YM}^2 N$ fixed. Using (6.17)(6.18), we find that

$$\begin{aligned}\tau_{p,q} &= 4\pi i \frac{e^{2\pi i/n}}{e^{2\pi i/n}-1} g_n \Lambda_0^{2n} u^{-1+1/n} N^2 \left[\eta^{2p} F(\eta^{2p}x) - \eta^{2q} F(\eta^{2q}x)\right] \\ &= 4\pi i \frac{e^{2\pi i/n}}{e^{2\pi i/n}-1} g_n \Lambda_0^{2n} u^{-1+1/n} N^2 \left[\left(F(x) + xF'(x)\right) 2\pi i \frac{p-q}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)\right], \qquad (7.3)\end{aligned}$$

where we set

$$x = \frac{4\Lambda_0^{2n}}{u^2}, \qquad F(x) \equiv F\left(\frac{n-1}{2n}, \frac{2n-1}{2n}, 2, x\right).$$

Note that we reinstated the parameter g_n that appears in the superpotential (3.9).

The large-N gauge theory is expected to have a dual description as a string theory with string coupling g_{st} proportional to $1/N$ [29]. By (7.3), we see that the tension of domain walls interpolating between adjacent vacua, where $(p-q)$ is of order N^0, is proportional to N or equivalently to $1/g_{st}$. This is the characteristic behaviour of a D-brane tension and it was in fact argued in [30] that the confining strings can end on the domain walls.

However, the expansion (7.3) is singular in the vicinity of the Argyres-Douglas points, $x = x_c = 1$, because

$$F(x) \approx A_1 + A_2(1-x)^{\frac{n+2}{2n}},$$

where the constants A_1 and A_2 are given in (4.16). Thus, $F'(x)$ is not defined for $x = x_c$. Actually, at the Argyres-Douglas points, the complexified tension becomes

$$\tau_{p,q} \approx N^{\frac{3n-2}{2n}} \left(p^{\frac{n+2}{2n}} - q^{\frac{n+2}{2n}}\right). \qquad (7.4)$$

The fractional power of N is clearly a signal of the breakdown of the large-N expansion. In fact, the $1/N$ corrections to (7.3) for $x \neq x_c$ read

$$\tau_{p,q} \approx N^2 A_2 (1-x)^{\frac{n+2}{2n}} \left[-2\pi i \left(\frac{n+2}{2n}\right) \frac{p-q}{N(1-x)} + \mathcal{O}\left(\frac{1}{(N(1-x))^2}\right)\right], \qquad (7.5)$$

which makes the singularity manifest.

7.1 The double scaling limit

Eq. (7.5) suggests that the divergences at $x = 1$ can be compensated by taking the limits $N \to \infty$ and $x \to x_c = 1$ in a correlated way as follows

$$x \to 1, \qquad N \to \infty, \qquad N(1-x) = cnst = \frac{1}{\kappa}. \qquad (7.6)$$

In particular, the rescaled tensions

$$\tau_{p,q}^{scaled} = (1-x)^{\frac{3n-2}{2n}} \tau_{p,q}, \qquad (7.7)$$

have a finite universal limit given by

$$\tau_{p,q}^{scaled} = \left(4\pi i \frac{e^{2\pi i/n}}{e^{2\pi i/n}-1} g_n \Lambda_0^{2n} u^{-1+1/n}\right) \frac{A_2}{\kappa^2} \left[(1-2\pi i p\kappa)^{\frac{n+2}{2n}} - (1-2\pi i q\kappa)^{\frac{n+2}{2n}}\right]. \qquad (7.8)$$

In a series of papers [9]–[12], Ferrari made a proposal to generalize the matrix model approach to non-critical strings [8] to the four dimensional case. The basic idea is to replace matrix integrals with four dimensional gauge theory path integrals with $N \times N$ adjoint Higgs fields. It was argued that a non-trivial low energy physics develops at special values of the Higgs vacuum expectation values or of the couplings of the Higgs potential. At these points the large-N expansion suffers from IR divergences. However, these divergences can be compensated by taking the limit $N \to \infty$ and approaching the critical points in a correlated manner. These double scaled theories are conjectured to be string theories. In particular, the continuous worldsheets are constructed from the large 't Hooft diagrams of the parent gauge theory.

These ideas were tested quantitatively on the moduli space of $\mathcal{N} = 2$ supersymmetric gauge theories in [10, 11]. It was shown that double scaling limits exist at Argyres-Douglas singularities. The non-trivial conformal field theories at these critical points are thus dual to a four dimensional string theory. In [6], the analysis was extended to the $\mathcal{N} = 1$ case. In particular, a $\mathrm{U}(N)$ gauge theory with cubic superpotential was studied and it was shown that there are critical values of the superpotential couplings where glueballs are massless, there are tensionless domain walls and confinement without a mass gap. At these critical points, the large-N expansion is singular and the tension of domain walls scales as a fractional power of N. Nevertheless, double scaling limits analogous to (7.6) exist and are then conjectured to define a four dimensional non-critical string theory.

The double scaling limits (7.6) fit into the above scenario and are consistent with the $\mathcal{N} = 2$ analysis of [11]. The conjecture is that they define an $\mathcal{N} = 1$ four dimensional non-critical string theory. In particular, the rescaled tensions (7.8) are the tensions of $D2$-branes in this string theory. In fact, in the weak coupling limit, the above tension goes like $1/\kappa$. It is interesting to note that going through the branch cuts of (7.8) the $D2$-brane is mapped into a soliton whose tension goes like $1/\kappa^2$ at weak coupling.

8. Conclusion

Using the techniques of [21], we constructed an $\mathcal{N} = 1$ theory with gauge group $\mathrm{U}(nN)$ and degree $n+1$ tree level superpotential whose matrix model spectral curve develops an Argyres-Douglas singularity. This theory is closely related to an underlying $\mathcal{N} = 2$ $\mathrm{U}(n)$ model. In fact, the one-dimensional parameter space of the $\mathrm{U}(nN)$ theory is actually isomorphic to a slice of the $\mathcal{N} = 2$ Coulomb moduli space of the $\mathrm{U}(n)$ theory: $n-1$ parameters of the $\mathrm{U}(n)$ Seiberg-Witten curve are set to zero and the remaining one parametrizes the most relevant deformation away from the singularity. The multiplication by N map [21, 24] allows to take the large-N limit while keeping the tree level superpotential fixed. In particular, only a finite, N-independent, number of parameters is adjusted. The calculation of the tension of domain walls in the $\mathrm{U}(nN)$ theory shows that the $1/N$ expansion breaks down at the Argyres-Douglas points. Nevertheless, it is possible to define appropriate double scaling limits (7.6) which are conjectured to yield the tension of 2-branes in the resulting $\mathcal{N} = 1$ four dimensional non-critical string theories as proposed by Ferrari in [9]–[13].

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A. Analytic continuation

Performing the analytic continuation of (4.6) we find

$$
\begin{aligned}
S_k =& \; C_{1,k}\, F\left(-\frac{1}{2}-\frac{1}{2n},\frac{1}{2}-\frac{1}{2n},\frac{1}{2},\frac{u^2}{4\Lambda^{2n}}\right)+C_{2,k}\, u\, F\left(-\frac{1}{2n},-\frac{1}{2n}+1,\frac{3}{2},\frac{u^2}{4\Lambda^{2n}}\right)\\
=& \; \frac{C_{1,k}\Gamma(1/2)}{\Gamma(1/2-1/2n)}\times\\
&\times\Bigg[\frac{(-u^2/4\Lambda^{2n})^{-1/2+1/2n}}{\Gamma(1+1/2n)}\times\\
&\quad\times\sum_{k=0}^{\infty}\left[\frac{(-1/2-1/2n)_{k+1}(-1/2n)_{k+1}}{k!(k+1)!}\left(\frac{4\Lambda^{2n}}{u^2}\right)^k\left(\log(-u^2/4\Lambda^{2n})+h_{1,k}\right)\right]+\\
&\quad+(-u^2/4\Lambda^{2n})^{+1/2+1/2n}\frac{\Gamma(1)}{\Gamma(1+1/2n)}\Bigg]+\\
&+\frac{C_{2,k}\Gamma(3/2)}{\Gamma(1-1/2n)}\,u\times\\
&\times\Bigg[\frac{(-u^2/4\Lambda^{2n})^{-1+1/2n}}{\Gamma(3/2+1/2n)}\times\\
&\quad\times\sum_{k=0}^{\infty}\Bigg[\frac{(-1/2-1/2n)_{k+1}(-1/2n)_{k+1}}{k!(k+1)!}\left(\frac{4\Lambda^{2n}}{u^2}\right)^k\times\\
&\qquad\times\left(\log\left(-\frac{u^2}{4\Lambda^{2n}}\right)+h_{2,k}\right)\Bigg]+\left(-\frac{u^2}{4\Lambda^{2n}}\right)^{1/2n}\frac{\Gamma(1)}{\Gamma(3/2+1/2n)}\Bigg].
\end{aligned}
$$

In the above expression there is a term proportional to $u^{1+1/n}$ that we need to set to zero. This determines $C_{2,k}$ as a function of $C_{1,k}$

$$
C_{2,k}=-\frac{e^{i\pi/2}}{2\Lambda^n}\frac{\Gamma(1-1/2n)\Gamma(3/2+1/2n)}{\Gamma(1/2-1/2n)\Gamma(1+1/2n)}\,C_{1,k}\,. \tag{A.1}
$$

We are left with

$$
\begin{aligned}
S_k =& \; \frac{C_{1,k}\Gamma(1/2)}{\Gamma(1/2-1/2n)}\frac{(-u^2/4\Lambda^{2n})^{-1/2+1/2n}}{\Gamma(1+1/2n)}\times\\
&\times\sum_{k=0}^{\infty}\frac{(-1/2-1/2n)_{k+1}(-1/2n)_{k+1}}{k!(k+1)!}\left(\frac{4\Lambda^{2n}}{u^2}\right)^k(h_{1,k}-h_{2,k})
\end{aligned}
$$

where

$$\begin{aligned}
h_{1,k} - h_{2,k} &= -\psi\left(\frac{1}{2} - \frac{1}{2n} + k\right) - \psi\left(\frac{1}{2n} - k\right) + \psi\left(-\frac{1}{2n} + 1 + k\right) + \psi\left(\frac{1}{2} + \frac{1}{2n} - k\right) \\
&= \left[\psi\left(\frac{1}{2} + \frac{1}{2n} - k\right) - \psi\left(\frac{1}{2} - \frac{1}{2n} + k\right)\right] - \left[\psi\left(\frac{1}{2n} - k\right) - \psi\left(-\frac{1}{2n} + 1 + k\right)\right] \\
&= \left[\pi \tan \pi\left(\frac{1}{2n} - k\right)\right] - \left[-\pi \cot \pi\left(\frac{1}{2n} - k\right)\right] = \frac{2\pi}{\sin(\pi/n)}\,.
\end{aligned}$$

Finally

$$\begin{aligned}
S_k = & \frac{2\pi}{\sin(\pi/n)} \frac{C_{1,k}\Gamma(1/2)}{\Gamma(1/2 - 1/2n)} \frac{(-u^2/4\Lambda^{2n})^{-1/2+1/2n}}{\Gamma(1+1/2n)} \times \\
& \times \sum_{k=0}^{\infty} \frac{(-1/2 - 1/2n)_{k+1}(-1/2n)_{k+1}}{k!(k+1)!} \left(\frac{4\Lambda^{2n}}{u^2}\right)^k .
\end{aligned}$$

Using

$$\begin{aligned}
\left(-\frac{1}{2} - \frac{1}{2n}\right)_{k+1} &\equiv \frac{\Gamma(-1/2 - 1/2n + k + 1)}{\Gamma(-1/2 - 1/2n)} = \frac{\Gamma(1/2 - 1/2n + k)}{\Gamma(1/2 - 1/2n)} \frac{\Gamma(1/2 - 1/2n)}{\Gamma(-1/2 - 1/2n)} \\
&= \left(\frac{1}{2} - \frac{1}{2n}\right)_k \frac{\Gamma(1/2 - 1/2n)}{\Gamma(-1/2 - 1/2n)}\,, \\
\left(-\frac{1}{2n}\right)_{k+1} &\equiv \frac{\Gamma(-1/2n + k + 1)}{\Gamma(-1/2n)} = \frac{\Gamma(1 - 1/2n + k)}{\Gamma(1 - 1/2n)} \frac{\Gamma(1 - 1/2n)}{\Gamma(-1/2n)} \\
&= \left(1 - \frac{1}{2n}\right)_k \frac{\Gamma(1 - 1/2n)}{\Gamma(-1/2n)}\,,
\end{aligned}$$

and

$$\Gamma(2)(2)_k = (2)_k = \Gamma(2+k) = (k+1)!$$

the expression is equivalent to

$$\begin{aligned}
S_k = & \frac{2\pi}{\sin(\pi/n)} \frac{C_{1,k}\Gamma(1/2)}{\Gamma(1/2 - 1/2n)} \frac{(-u^2/4\Lambda^{2n})^{-1/2+1/2n}}{\Gamma(1+1/2n)} \frac{\Gamma(1/2 - 1/2n)}{\Gamma(-1/2 - 1/2n)} \frac{\Gamma(1 - 1/2n)}{\Gamma(-1/2n)} \times \\
& \times \sum_{k=0}^{\infty} \frac{(1/2 - 1/2n)_k (1 - 1/2n)_k}{k!\,(2)_k} \left(\frac{4\Lambda^{2n}}{u^2}\right)^k \\
= & \frac{2\pi}{\sin(\pi/n)} \frac{\sqrt{\pi}C_{1,k}}{\Gamma(1+1/2n)} \frac{(-u^2/4\Lambda^{2n})^{-1/2+1/2n}}{\Gamma(-1/2 - 1/2n)} \frac{\Gamma(1 - 1/2n)}{\Gamma(-1/2n)} \times \\
& \times F\left(\frac{1}{2} - \frac{1}{2n}, 1 - \frac{1}{2n}, 2, \frac{4\Lambda^{2n}}{u^2}\right)
\end{aligned}$$

Comparing with (4.8), we find

$$C_{3,k} = e^{i\pi/2 + i\pi/2n} \frac{2\pi}{\sin(\pi/n)} \left(\frac{\sqrt{\pi}\, 2^{\frac{n-1}{n}}}{2n\Gamma(1+1/2n)\Gamma(-1/2 - 1/2n)}\right) \Lambda^{n-1} C_{1,k}\,. \tag{A.2}$$

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